Midterm 1: MAT 319 and MAT 320

Instructions: Complete all problems below. You may not use calculators or other aides, including cell phones and books. Show all of your work. **Be sure to write your name and student ID on each page that you hand in.**

1.(20pts) Let

$$a_n = \left(\frac{4+2(-1)^n}{5}\right)^n.$$

Compute the limsup and liminf of $|a_n|^{1/n}$ and $|a_{n+1}/a_n|$.

Obsere that

$$\frac{2}{5} \le |a_n|^{1/n} = \frac{4+2(-1)^n}{5} \le \frac{6}{5}.$$

Since the upper & lower bounds are achieved infinitely often

$$\limsup |a_n|^{1/n} = \frac{6}{5}, \quad \liminf |a_n|^{1/n} = \frac{2}{5}.$$

Next observe that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left(\frac{4+2(-1)^{n+1}}{5}\right)^{n+1}}{\left(\frac{4+2(-1)^n}{5}\right)^n} = \left(\frac{4+2(-1)^{n+1}}{5}\right)\left(\frac{2-(-1)^n}{2+(-1)^n}\right)^n$$

Since $\frac{4+2(-1)^{n+1}}{5}$ stays inside $(\frac{2}{5}, \frac{6}{5})$ and

$\left(\frac{2-(-1)^n}{2+(-1)^n}\right)^n \to \infty$ for $n =$ odd we find $\limsup \left|\frac{a_{n+1}}{a_n}\right| = \infty$.

Since $\left(\frac{2-(-1)^n}{2+(-1)^n}\right)^n \to 0$ for $n =$ even we find $\liminf \left|\frac{a_{n+1}}{a_n}\right| = 0$.

2.(20pts) The Fibonacci sequence F_n is defined inductively by

- $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$,
- $F_0 = 0$, and $F_1 = 1$.

Thus $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on.

a) Show that there are constants α and β for which

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ for } n \geq 0 .$$

b) Determine if the following series of Fibonacci number reciprocals converges:

$$\sum_{n=0}^{+\infty} \frac{1}{F_n}.$$

a) To find α, β solve $n=1$, $n=2$ ie. $\frac{\alpha-\beta}{\sqrt{5}} = F_1 = 1$, $\frac{\alpha^2-\beta^2}{\sqrt{5}} = F_2 = 1$.

Get $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Claim: $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ for all $n \geq 0$.

Proof: Use induction. Clearly it is true for $n=0,1$. Assume it is true for n and prove it for $n+1$:

$$F_{n+1} = F_n + F_{n-1} = \frac{\alpha^n - \beta^n}{\sqrt{5}} + \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}$$

$$= \frac{\alpha^n(1+\frac{1}{\alpha}) - \beta^n(1+\frac{1}{\beta})}{\sqrt{5}} = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}}$$

$\square$

Since $1 + \frac{1}{\alpha} = \alpha$, $1 + \frac{1}{\beta} = \beta$.

b) $$\sum_{n=0}^{\infty} \frac{1}{F_n} = \sum_{n=0}^{\infty} \frac{\sqrt{5}}{\alpha^n - \beta^n} = \sum_{n=0}^{\infty} \frac{1}{\alpha^n} \cdot \frac{\sqrt{5}}{1-(\frac{\beta}{\alpha})^n}$$

Since $\left|\frac{\beta}{\alpha}\right| < 1$ we have $\frac{\sqrt{5}}{1-(\frac{\beta}{\alpha})^n} \leq 2\sqrt{5}$ for n large.

Hence the series converges by the comparison test where the comparison series is the geometric series $\sum_{n=0}^{\infty} \frac{2\sqrt{5}}{\alpha^n} = \frac{2\sqrt{5}}{1-\alpha^{-1}}$.

3.(20pts) Let a_n be a sequence. Suppose that $a_n \neq 0$ for all n, and that the limit $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists.

a) Prove that if $L < 1$ then $\lim_{n\to\infty} a_n = 0$.

b) Prove that if $L > 1$ then $\lim_{n\to\infty}|a_n| = +\infty$.

Using the definition of limit, we may choose $\varepsilon > 0$ so small that there exists an N with the property

$$\begin{cases} \left|\frac{a_{n+1}}{a_n}\right| \leq \alpha < 1 & \text{if } L<1 \\ \left|\frac{a_{n+1}}{a_n}\right| \geq \beta > 1 & \text{if } L>1 \end{cases} \quad \text{for all } n \geq N.$$

a) Take any $n > N$ and observe that

$$|a_n| = \left|\frac{a_n}{a_{n-1}}\right| \cdots \left|\frac{a_{N+1}}{a_N}\right| \leq \alpha^{n-N}.$$

Hence $\left|\lim_{n\to\infty} a_n\right| = \lim_{n\to\infty} |a_n| \leq \alpha^{-N} \lim_{n\to\infty} \alpha^n = 0$.

b) In this case the same expression yields $|a_n| \geq \beta^{n-N}$.

Hence $\lim_{n\to\infty} |a_n| \geq \lim_{n\to\infty} \beta^{n-N} = \infty$ which implies that $\lim_{n\to\infty} |a_n| = \infty$.

$\square$

4.(20pts) Let a_n be a sequence of nonnegative numbers ($a_n \geq 0$ for all $n \geq 0$). Show that we have

$$\sum_{n=0}^{+\infty} a_n = \sup \left\{ \sum_{j \in J} a_j \mid J \subset \mathbb{N} \text{ finite subset} \right\} .$$

Define $A = \{ \sum_{j \in J} a_j \mid J \subset \mathbb{N}$ finite subset$\}$. By definition of the sum, $\sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} S_N$ where $S_N = \sum_{n=0}^{N} a_n$.

Since $S_N \in A$ for each N, we have $\lim_{N \to \infty} S_N \leq \sup A$ and hence $\sum_{n=0}^{\infty} a_n \leq \sup A$. On the other hand $\sum_{n=0}^{\infty} a_n$ is an upper bound for A since (by $a_n \geq 0$) $\sum_{n=0}^{\infty} a_n \geq \sum_{j \in J} a_j$ for any J. Hence, by definition of sup we have $\sup A \leq \sum_{n=0}^{\infty} a_n$. It follows that $\sum_{n=0}^{\infty} a_n = \sup A$. $\square$

5.(20pts) Let f and g be two functions with domain $\mathbb{R}$, such that $f(x) < g(x)$ for all $x \in \mathbb{Q}$. Prove that if f and g are continuous then the inequality $f(x) \leq g(x)$ holds for **all** $x \in \mathbb{R}$.

Let $h(x) = g(x) - f(x)$. Then this is equivalent to the following. Claim: Let h be continuous on $\mathbb{R}$ with $h(x) > 0$ for all $x \in \mathbb{Q}$, then $h(x) \geq 0$ for all $x \in \mathbb{R}$.

Proof: Suppose $\exists\, x_0 \in \mathbb{R}$ with $h(x_0) < 0$. There is a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ with $\lim_{n\to\infty} x_n = x_0$. Since h is continuous we have $\lim_{n\to\infty} h(x_n) = h(x_0) < 0$. On the other hand since $x_n \in \mathbb{Q}$ we have $h(x_n) > 0$ and hence $\lim_{n\to\infty} h(x_n) \geq 0$, otherwise there must exist some $h(x_n) < 0$ which is not possible. This is a contradiction. We conclude that $h(x) \geq 0$ for all $x \in \mathbb{R}$. $\square$